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SHELL VERSUS ARCH ACTION IN BARREL SHELLS

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STRUCTURAL DIVISION

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SHELL VERSUS ARCH ACTION IN BARREL SHELLS

M. G. Salvadori,[1] M. ASCE

INTRODUCTION

The essential difference in structural behavior between thin cylindrical shells supported by stiffening ribs (arches), and cylindrical shells resting directly on the ground is that the first type transmits loads to the stiffeners almost entirely by direct stresses, while the second develops sizable bending stresses in carrying the load to the ground. The transmission of loads by a thin shell to the stiffeners from which it hangs is usually called "shell action," while the behavior of the unstiffened shell resting on the ground is called "arch action" inasmuch as each slice of shell behaves like a separate arch.

When a cylindrical shell stiffened by transverse ribs is built-in into a foundation, loads are carried to the ground by a combination of both actions and it is economical to know: 1) What part of the load is carried by shell action, and 2) how large are the bending moments created by arch action.

It is the purpose of this paper to evaluate these two factors for the particularly simple case of: a) a cylindrical shell of arbitrary cross-section: b) with vertical tangents along its longitudinal boundaries: c) supported by infinitely rigid stiffeners: d) rigidly built-in into the foundation: e) under a uniformly distributed load.

This study was made possible by the recent publication of the Manual of Engineering Practice No. 31 "Design of Cylindrical Concrete Shell Roofs" of the American Society of Civil Engineers, which contains tables for the evaluation of displacements and stresses in circular cylindrical shells due to loads applied to the shell boundaries. Without the help of these tables it would be practically impossible to consider problems of this type, in view of the extremely burdensome calculations involved. It is hoped that this application of the Manual will indicate its wide usefulness and increase its popularity among designing engineers.

The structural results obtained in this study are summarized in the last section of the paper.

Membrane Stresses and Displacements in Cylindrical Shells of Arbitrary Cross-Section

With the symbols of Fig. 1, the direct membrane stresses in a shell satisfy the three differential equations:[2]

$$\frac{\partial N_x}{\partial x} + \frac{1}{r}\frac{\partial N_{x\phi}}{\partial \phi} = -X \tag{1a}$$

1. Prof. of Civ. Eng., Columbia Univ., New York, N.Y.
2. See, for example, S. Timoshenko, "Theory of Plates and Shells," McGraw-Hill Book Company, New York, 1940, p. 384.

$$\frac{\partial N_{x\phi}}{\partial x} + \frac{1}{r}\frac{\partial N_{\phi}}{\partial \phi} = -Y \tag{1b}$$

$$N_{\phi} = -Zr, \tag{1c}$$

while the displacements u, v, w in the longitudinal, tangential and radial directions (positive as shown in Fig. 1) are given by the solution of the following equations:[3]

$$\frac{\partial u}{\partial x} = \frac{N_x}{Eh} \tag{2a}$$

$$\frac{1}{r}\frac{\partial v}{\partial \phi} + \frac{w}{r} = \frac{N_{\phi}}{Eh} \tag{2b}$$

$$\frac{\partial v}{\partial x} + \frac{1}{r}\frac{\partial u}{\partial \phi} = \frac{2N_{x\phi}}{Eh} \tag{2c}$$

In Eqs. (1) and (2):

X, Y, and Z are the components of the load in the longitudinal, tangential and normal directions per unit area of shell;

r, the radius of the shell cross-section at the point considered, is a function of ϕ;

E is Young's modulus;

h is the shell thickness either constant or, at least, constant in the neighborhood of the longitudinal shell boundary;

Poisson's ratio ν has been taken equal to zero, as is usual in dealing with reinforced concrete shells.

The load considered is vertical and constant across the shell, and varies cosinusoidally in the longitudinal direction; hence:

$$X = 0, \quad Y = p \sin\phi \cos\phi \cos \pi x/L, \qquad Z = p \cos^2\phi \cos \pi x/L \tag{3}$$

Direct integration of Eqs. (1) with the load components of Eqs. (3) under the assumption that:

$$N_{x\phi}\,\big]_{x=0} = 0\ ; \qquad N_x\,\big]_{x=\pm L/2} = 0, \tag{4}$$

gives the membrane stresses:

$$N_{\phi} = -p\,r \cos^2\phi \cos \pi x/L \tag{5a}$$

$$N_{x\phi} = \frac{1}{\pi}\, p\, r_o \left(\frac{L}{r_o}\right) F(\phi) \sin \pi x/L \tag{5b}$$

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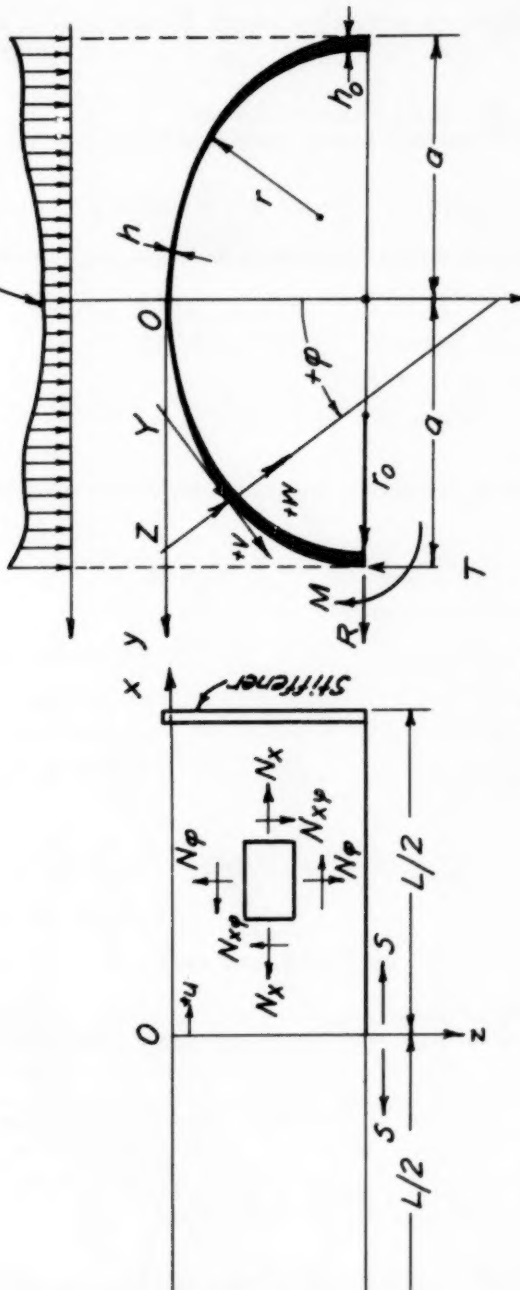
3. See, for example, the Manual of Engineering Practice No. 31 of the American Society of Civil Engineers "Design of Cylindrical Concrete Shell Roofs," p. 117. The manual is obtainable from the Headquarters of the Society, 29 West 39th Street, New York, N.Y.



p = p(y)
h
r
a
a
h_o
O
Y
Z
z
+φ
+w
+v
r_o
M
R
T
y
x
Stiffener
Stiffener
N_x
N_xφ
N_φ
u
S
S
L/2
L/2

Fig. 1

$$N_x = \frac{1}{\pi^2} p\, r_o \left(\frac{L}{r_o}\right)^2 \left(\frac{r_o}{r}\right) F' \cos \pi x/L. \tag{5c}$$

In Eqs. (5):
r_0 is the value of r at the longitudinal edge;
primes stand for derivations with respect to ϕ;

and:
$$F(\phi) = \frac{r'}{r} \cos^2 \phi - \frac{3}{2} \sin 2\phi \tag{6}$$

When the shell has vertical tangents at the edge, i.e., when $\phi = \pi/2$ at the edge:

$$N_\phi]_{\phi=\pi/2} = 0 \;; \quad N_{x\phi}]_{\phi=\pi/2} = 0 \;;$$
$$N_x]_{\phi=\pi/2} = \frac{3}{\pi^2} p\, r_o \left(\frac{L}{r_o}\right)^2 \cos \pi x/L. \tag{5d}$$

Substitution of the stresses of Eq. (5) in Eqs. (2) and integration under the assumption that:

$$u\,]_{x=o} = 0 \;; \quad v\,]_{x=\pm L/2} = 0 \,, \tag{7}$$

gives the following displacements:

$$u = \frac{1}{\pi^3} \frac{pr_o^2}{Eh} \left(\frac{L}{r_o}\right)^3 \left(\frac{r_o}{r}\right) F' \sin \pi x/L \tag{8a}$$

$$v = -\frac{1}{\pi^2} \frac{pr_o^2}{Eh} \left(\frac{L}{r_o}\right)^2 \left[2F - \frac{1}{\pi^2}\left(\frac{L}{r_o}\right)^2 \frac{r_o}{r}\left(\frac{r_o}{r} F'\right)'\right] \cos \pi x/L \tag{8b}$$

$$w = \frac{pr_o^2}{Eh} \left\{ -\frac{1}{\pi^2}\left(\frac{L}{r_o}\right)^2 \left[2F' - \frac{1}{\pi^2}\left(\frac{L}{r_o}\right)^2 \left(\frac{r_o}{r}\left(\frac{r_o}{r} F'\right)'\right)'\right] + \right.$$
$$\left. + \left(\frac{r}{r_o}\right)^2 \cos^2 \phi \right\} \cos \pi x/L, \tag{8c}$$

where h is assumed constant.

The rotation of the edge, <u>when the tangent to the edge is vertical</u>, is given by:

$$\theta_o = \frac{1}{r_o} \frac{\partial w}{\partial \phi}\Big]_{\phi=\pi/2} =$$

$$\frac{pr_o}{Eh_o} \left\{ -\frac{1}{\pi^2}\left(\frac{L}{r_o}\right)^2 \left[2F'' - \frac{1}{\pi^2}\left(\frac{L}{r_o}\right)^2 \left(\frac{r_o}{r}\left(\frac{r_o}{r} F'\right)'\right)''\right] + \right.$$
$$\left. + 2\frac{r\,r'}{r_o^2} \cos^2 \phi - \left(\frac{r}{r_o}\right)^2 \sin 2\phi \right\}_{\phi=\pi/2} \cos \pi x/L. \tag{9}$$

The values of the displacements at the longitudinal edge $\phi = \pi/2$ shall be indicated by a subscript zero. Making $\phi = \pi/2$ in Eqs. (8, 9) and noting that

$$r'_o = \frac{dr}{d\phi}\Big]_{\phi=\pi/2} = 0 \tag{10}$$

since the tangent to the shell is vertical at $\phi = \pi/2$, we obtain by means of Eq. (6):

$$u_o = u\Big]_{\phi=\pi/2} = \frac{3}{\pi^3}\frac{pr_o^2}{Eh_o}\left(\frac{L}{r_o}\right)^3 \sin \pi x/L \tag{11a}$$

$$v_o = v\Big]_{\phi=\pi/2} = 0 \tag{11b}$$

$$w_o = w\Big]_{\phi=\pi/2} = -\frac{3}{\pi^2}\frac{pr_o^2}{Eh_o}\left(\frac{L}{r_o}\right)^2 \left\{ 2 - \frac{1}{\pi^2}\left(\frac{L}{r_o}\right)^2 \left[\frac{r''_o}{r_o} - 4\right]\right\} \cos \pi x/L \tag{11c}$$

$$\theta_o = \frac{9}{\pi^4}\frac{pr_o}{Eh_o}\left(\frac{L}{r_o}\right)^4 \frac{r'''_o}{r_o} \cos \pi x/L \tag{11d}$$

These expressions for the displacements may be proved to be also valid for a variable h, provided $\frac{dh}{d\phi}\Big]_{\phi=\pi/2} = 0$.

Hence, if the shell cross-section is such that, in addition to condition (10), the shell radius satisfies the condition

$$r'''_o = \frac{d^3 r}{d\phi^3}\Big]_{\phi=\pi/2} = 0, \tag{12}$$

the rotation at the edge is also zero:

$$\theta_o = 0 \, . \tag{13}$$

Thus the shell edge has no vertical displacements ($v_0 = 0$), nor does it rotate, but only moves outward and longitudinally.

It is interesting to notice that these properties hold even if the load p varies across the shell as a function of y (Fig. 1), provided p be not infinite at the edge (as is the case for the dead load), since all the derivatives of p with respect to ϕ are computed by means of the operator:

$$\frac{d}{d\phi}\Big]_{\phi=\pi/2} = \frac{d}{dy}\frac{dy}{d\phi}\Big]_{\phi=\pi/2} \tag{14}$$

and $dy/d\phi\,\Big]_{\phi=\pi/2} = 0$, since the shell is vertical at the edge.

Conditions (10) and (12) are satisfied by all cross-sectional shapes symmetrical about the horizontal diameter y, and in particular by circular and elliptical sections.

Boundary Reactions at Shell Edges

When a cylindrical shell is built-in at the longitudinal edges statically indeterminate reactions per unit length (Fig. 1):

$T \cos \pi x/L$ in the tangential (vertical) z-direction,
$R \cos \pi x/L$ in the radial (horizontal) y-direction,
$S \sin \pi x/L$ in the longitudinal x-direction,
$M \cos \pi x/L$ in the ϕ-direction,

arise capable of producing displacements equal and opposite to those due to the load.

The displacements due to these boundary reactions have been evaluated in the Manual of Engineering Practice No. 31, quoted in the Introduction, for the case of circular cylinders; but, since these displacements are limited to the immediate neighborhood of the boundary, they are good approximations of the displacements in a shell of arbitrary cross-section provided the radius of the approximating circular cylinder be taken equal to the radius of curvature r_o of the shell at the boundary.

To determine the boundary reactions we shall assume that the load p on the shell is uniform and shall expand p into a Fourier series in the x direction:

$$p = \frac{4}{\pi} p \sum_{n=1,3,5,\ldots} \frac{1}{n} \cos n \pi x/L . \tag{15}$$

If we take into account the first term of this series only, the load distribution becomes

$$p(x) = \frac{4}{\pi} p \cos \pi x/L \tag{15a}$$

and the displacements due to $p \cos \pi x/L$ [Eqs. (11)] must all be multiplied by $4/\pi$. It may be checked that the error made by dropping all the terms of the series (15) but the first is admissible for a problem of the kind here considered.

By means of the coefficients in Table 3b (case n = 1) of Manual No. 31 the following set of 4 equations for the 4 unknown reactions T, R, S and M may be easily written:

$$\begin{aligned} \frac{Eh_o}{L} \Sigma v = a_{11} T + a_{12} RL/h_o + a_{13} S \\ + a_{14} M/h_o + \frac{4Eh_o}{\pi L} v_{o,o} = 0 \end{aligned} \tag{16a}$$

$$\begin{aligned} \frac{EI_o}{Lh_o} \Sigma \theta = a_{21} T + a_{22} RL/h_o + a_{23} S \\ + a_{24} M/h_o + \frac{4EI_o}{\pi L h_o} \theta_{o,o} = 0 \end{aligned} \tag{16b}$$

$$\frac{Eh_o^2}{L^2} \Sigma\, w = a_{31}\, T + a_{32}\, RL/h_o + a_{33}\, S$$

$$+ a_{34}\, M/h_o + \frac{4Eh_o^2}{\pi L^2}\, w_{o,o} = 0 \tag{16c}$$

$$Eh_o\, \Sigma \frac{\partial u}{\partial x} = a_{41}\, T + a_{42}\, RL/h_o + a_{43}\, S$$

$$+ a_{44}\, M/h_o + \frac{4}{\pi}\, N_x \Big]_{\substack{\phi=\pi/2\\ x=L/2}} = 0 \tag{16d}$$

where $I_o = h_o^3/12$.

Equation (16d) actually states that the total strain in the longitudinal direction, $\frac{\partial u}{\partial x}$, is zero on the boundary, rather than u itself. But since u is zero at x = 0, these two conditions are equivalent. Moreover the multiplying factor Eh_o, by Eq. (2a) changes the strain due to the load into the stress N_x, and the corresponding terms due to the statically indeterminate reactions into the stresses N_x due to the reactions. These last stresses are evaluated by means of the coefficients in Table 3A of Manual No. 31.

Under assumptions (10) and (12), and with N_x given by Eq. (5d) and w_o by Eq. (11c), the constants of Eqs. (16c, d) become:

$$\frac{4Eh_o^2}{\pi L^2}\, w_o\Big]_{x=L/2} = -\frac{24}{\pi^5}\left(\frac{r_o h_o}{L^2}\right)\left[2\left(\frac{L}{r_o}\right)^2 + \pi^2\right] p\, r_o \tag{17a}$$

$$\frac{4}{\pi}\, N_x\Big]_{\substack{\phi=\pi/2\\ x=L/2}} = \frac{12}{\pi^2}\, p\, r_o \left(\frac{L}{r_o}\right)^2 \tag{17b}$$

where r_o''/r_o is taken equal to zero in Eq. (11c), since the cross-section of the cylinder is approximated by a circle of radius r_o.

Solution of Eqs. (16) with the constants of Eqs. (17) gives the statically indeterminate reactions R, S, T and M.

Vertical Reactions and Boundary Moments

In order to evaluate the vertical reactions and the bending moments at the shell boundary for the most commonly encountered shell characteristics, Eqs. (16) have been solved for the values of the parameters $r_o h_o/L^2$ and L/r_o of Table I.[4]

The results are presented in non-dimensional form in Table I, where:

$$\left(\frac{a}{r_o}\right) t\,\% = 100\, \frac{(2/\pi)\, TL}{p\, r_o L} \tag{18}$$

4. The computations for this table were performed by Mr. P. J. O'Leary, Research Associate, Department of Civil Engineering, Columbia University.

is the percentage ratio of the total vertical shell reaction to the total arch reaction, i.e., the percentage of load carried directly by the ground, and:

$$\left(\frac{a}{r_o}\right)^2 m^o/o = \frac{M}{M_a} = \frac{M}{(4/3 - \pi^2/8)\ p\ r_o^2/(\pi^2/2 - 4)} \tag{19}$$

is the percentage ratio of the maximum shell moment M (at x = 0) to the moment M_a in a circular arch uniformly loaded.[5] These ratios are valid for circular shells only; for shells of span 2a, t^o/o must be multiplied by r_o/a and m^o/o by $(r_o/a)^2$.

The following conclusions may be drawn from the results of Table I.

a) For small values of the ratio L/r_o, say, for $L/r_o < 1$ the load is transmitted almost entirely by the shell to the stiffeners.

It will be noticed that for very thin shells (small values of r_oh_o/L^2) the vertical reaction of the ground is negative: this means that the reaction is downward and that the foundation pulls on the shell. These negative reactions are required essentially by the local displacements at the boundary and not by a downward deflection due to the load. The stiffening effect of the boundary shear S, which produces high tensile stresses N_x, is essentially responsible for this result.

For large values of r_oh_o/L^2, i.e., for thick shells, the foundation reaction is always positive (upward), although negligible for values of $L/r_o < 0.5$.

In practice the ratio L/r_o is fairly small. For large shells spanning over 200 ft. L/r_o is usually smaller than 0.4 and is often less than 0.3. In the United States L/r_o is taken equal to 0.20 - 0.23 for most largeshells. Hence, for these shells the foundation reaction, if any, is negligible.

As L/r_o increases beyond 1 the reaction of the foundation increases rapidly. The last column of Table I indicates the value of L/r_o (obtained by graphical extrapolation) for which the reaction becomes equal to the arch reaction, i.e., for which the whole load is carried by the foundation. For very thin shells this ratio is as high as 5.21, but for thick shells it goes down to 1.79.

Values of t^o/o greater than $100^o/o$ indicate that the local displacements at the boundary require very high vertical reactions and that, in this case, the action of the shell on the stiffener would be reversed.

It must be remembered that in deriving the value of the reaction the stiffeners were considered perfectly rigid. In practice the stiffeners will deflect and the shell edge will move. Therefore, the vertical reaction of the foundation may be larger than indicated in Table I. Thus, as a consequence of the elasticity of the stiffeners, the foundation reaction may carry the whole load for ratios L/r_o smaller than those indicated in the last column of the Table.

For the case of a non-circular shell it should be remembered that t^o/o must be multiplied by r_o/a, where a is half the shell span. For example, for an elliptical shell of $r_oh_o/L^2 = 0.015$ and of semiaxes a = 100 ft. and b = 50 ft.:

$$r_o = \frac{b^2}{a} \quad ; \quad r_o/a = \frac{b^2}{a^2} = 0.25$$

5. See, for instance, M. G. Salvadori and A. D. Ateshoglou, "Ribless Cylindrical Shells," Journal of the American Concrete Institute, 1955.

TABLE I

$\frac{r_o h_o}{L^2}$	$\frac{L}{r_o}$ =	0.2	0.5	1.0	1.5	2.0	2.5	3.0	4.0		$\frac{L}{r_o}$
0.002	$(\frac{a}{r_o})t^o/o$	-0.127	-0.779	-2.93	-5.87	-8.68	-10.02	-8.18	13.69	$(\frac{a}{r_o})t^o/o=100$	5.21
	$(\frac{a}{r_o})^2 m^o/o$	-0.000	- .003	-0.012	0.120	0.479	2.83	27.32	65.78	$(\frac{a}{r_o})^2 m^o/o=100$	4.21
0.006	$(\frac{a}{r_o})t^o/o$	-0.119	-0.710	-2.39	-3.68	-2.32	4.85	21.91		$(\frac{a}{r_o})t^o/o=100$	4.01
	$(\frac{a}{r_o})^2 m^o/o$	0.000	0.009	0.212	1.61	7.43	25.56	71.76		$(\frac{a}{r_o})^2 m^o/o=100$	3.22
0.015	$(\frac{a}{r_o})t^o/o$	-0.049	-0.237	-0.032	3.37	14.56	39.97			$(\frac{a}{r_o})t^o/o=100$	3.09
	$(\frac{a}{r_o})^2 m^o/o$	0.002	0.065	1.26	8.27	34.44	110.16			$(\frac{a}{r_o})^2 m^o/o=100$	2.36
0.040	$(\frac{a}{r_o})t^o/o$	0.143	1.015	5.83	19.77	51.70				$(\frac{a}{r_o})t^o/o=100$	2.39
	$(\frac{a}{r_o})^2 m^o/o$	0.007	0.307	5.72	35.81	143.45				$(\frac{a}{r_o})^2 m^o/o=100$	1.86
0.100	$(\frac{a}{r_o})t^o/o$	0.444	2.98	14.87	44.52					$(\frac{a}{r_o})t^o/o=100$	1.79
	$(\frac{a}{r_o})^2 m^o/o$	0.027	1.10	20.21	124.62					$(\frac{a}{r_o})^2 m^o/o=100$	1.27

and the shell reaction for $L/r_o = 3.09$ is 25°/o of the total load rather than 100°/o.

b) For small values of L/r_o, say, less than 0.5 the maximum bending moment at the shell boundary is a small fraction of the arch moment. Moreover, for very thin shells the moment has a negative sign, producing tensile stresses on the outer fibers of the shell. For the average value of $r_o h_o/L^2$ encountered in practice the moment in the shell is positive and increases rapidly with L/r_o, the faster the thicker the shell. The last column of Table I gives the ratio L/r_o (obtained by graphical extrapolation) for which the maximum moment becomes equal to the arch moment: this ratio decreases from 4.21 for very thin shells to 1.27 for thick shells. The reduction factor in the moment due to the ratio r_o/a for non-circular shells is $(r_o/a)^2$; thus, for the elliptical shell of the previous example the moment at $L/r_o = 2.5$ would only be

$$m\ {}^{o}/o = 110.16 \times (.25)^2 = 6.89\ {}^{o}/o$$

instead of 110.16 °/o. It must be noticed that the reduction factor r_o/a may be larger than one. Thus, for instance, in so-called conoidal sections (circular sections with center off the axis of the shell) r_o/a may well be as high as 1.5.

The values of the bending moments are critical in the design of thin shells, since they determine its thickness in many practical cases. Remembering that the elasticity of the stiffeners may increase the ratio m, it is apparent from Table I that for shells of circular or almost circular shape built-in into a foundation it is uneconomical to choose ratios L/r_o larger than unity and that, in fact, L/r_o is wisely chosen smaller than 1/2.

c) The conclusions reached on the basis of the results of Table I apply only to a uniform vertical load and to shells with vertical tangents stiffened by rigid arches. It would be unsafe to apply these results literally to other types of shells under different conditions. But the engineer, with his keen physical intuition, may well use these conclusions as a guide in the economical design of cylindrical thin shells, provided he does not forget on the one hand the many assumptions on which this study is based, and on the other the numerous essential factors, here neglected, influencing shell design.